

NASA TM X-

REPRESENTATION OF MAGNETIC FIELDS IN SPACE

(NASA-TM-X-70847) REPRESENTATION OF
MAGNETIC FIELDS IN SPACE (NASA) 59 p HC
CSCL 03B

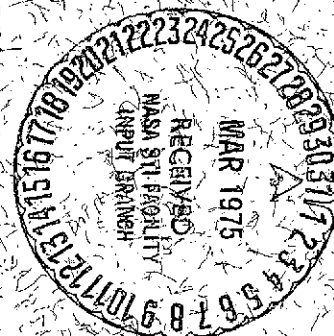
N75-18824

Unclas

G3/46 13275

DAVID P. STERN

MARCH 1975



GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

Reproduced by
**NATIONAL TECHNICAL
INFORMATION SERVICE**
US Department of Commerce
Springfield, VA. 22151

PRICES SUBJECT TO CHANGE

For information concerning availability
of this document contact:

Technical Information Division, Code 250
Goddard Space Flight Center
Greenbelt, Maryland 20771

(Telephone 301-982-4488)

"This paper presents the views of the author(s), and does not necessarily
reflect the views of the Goddard Space Flight Center, or NASA."

Representation of Magnetic Fields in Space

David P. Stern
Theoretical Studies Group
Goddard Space Flight Center
Greenbelt, Maryland 20771

To be submitted to Reviews of Geophysics and Space Physics

Representation of Magnetic
Fields in Space

INTRODUCTION

- (1) THE VECTOR POTENTIAL \underline{A}
- (2) THE SCALAR POTENTIAL γ
 - (a) Introduction
 - (b) Spherical Harmonics
 - (c) Geomagnetic Models
 - (d) Current-free Models of the Magnetosphere
 - (e) Change of Coordinates
 - (f) Practical Problems related to Uniqueness
- (3) TOROIDAL AND POLOIDAL FIELDS
 - (a) Introduction
 - (b) Vector Potential for Curl-free Fields
 - (c) Models of the Magnetosphere
 - (d) Spherical Vector Harmonics
- (4) EULER POTENTIALS
 - (a) Definition and General Properties
 - (b) Examples
 - (c) The L Parameter
- (5) LOCAL REPRESENTATIONS

AFTERWORD

INTRODUCTION

The purpose of this work is to review several methods by which a magnetic field \underline{B} in space can be represented, with particular attention to problems of the observed geomagnetic field. Time dependence will be assumed to be negligible and five main classes of representation will be described, as follows:

- (1) Representation by a vector potential \underline{A} :

$$\underline{B} = \nabla \times \underline{A} \quad (1)$$

This is a general form to which other forms noted here can always be converted.

- (2) Representation by a scalar potential γ :

$$\underline{B} = - \nabla \gamma \quad (2)$$

This representation is available only in current-free regions and it is particularly appropriate to the main geomagnetic field, where the expansion of γ in spherical harmonics is of considerable interest.

- (3) Representation by orthogonal vectors, in particular those related to spherical coordinates

$$\underline{B} = \nabla \times \Psi_1 \underline{r} + \nabla \times \nabla \times \Psi_2 \underline{r} \quad (3)$$

This representation is related to spherical vector harmonics; it has been used in dynamo theory and in a variety of problems. As will be seen, it is also useful in devising models of the earth's magnetosphere field.

(4) Representation by Euler potentials

$$\underline{B} = \nabla\alpha \times \nabla\beta \quad (4)$$

This is the only representation which includes explicit information about the configuration of magnetic field lines, but its nonlinear character makes its derivation difficult. In the earth's magnetic field α is related to McIlwain's L parameter which is useful in the study of the motion of trapped particles.

(5) Local representations, in which \underline{B} is expanded around its value at some reference point

$$\underline{B} = B_0 + \underline{r} \cdot \nabla B_0 + \frac{1}{2} \underline{r} \underline{r} : \nabla \nabla B_0 + \dots \quad (5)$$

Such expansions find use in the theory of guiding center motion and in describing the vicinity of neutral points at which \underline{B} vanishes.

The present discussion is not intended to be self-contained: matters on which recent reviews exist in the general literature will only be briefly described with references directing the reader to more elaborate treatments. References will also be given to articles in which various mathematical tools described here are employed or mentioned, but the mentioning of such articles does not always imply concurrence with all conclusions expressed there, nor is the review of the literature meant to be complete. In some cases review papers are cited mainly in order that they may provide the reader with a guide to earlier work not mentioned here.

(1) THE VECTOR POTENTIAL A

The representation (1) is the most general one and others may be reduced to it. For instance, (3) leads naturally to

$$\underline{A} = \psi_1 \underline{r} + \nabla \times \psi_2 \underline{r} \quad (6)$$

while (4) gives

$$\underline{A} = \alpha \nabla \beta \quad (7)$$

which is orthogonal to B. The representation (2) also has an equivalent form (1) and if γ is expanded in spherical harmonics an equivalent representation of the form (3) is readily obtained (as will be shown), allowing (6) to be used. All these choices of A are indeterminate within the gradient of an arbitrary scalar ψ_0 , since the addition of such a gradient to A does not affect (1).

The representation (1) is part of the standard treatment found in practically all texts on classical electrodynamics and will therefore not be discussed in detail. Its main usefulness, in that form, is that if the current density j creating the field is everywhere known, A is readily derived (assuming B to be a vacuum field) by volume integration

$$\underline{A}(\underline{r}) = (\mu_0/4\pi) \int \frac{j(\underline{r}')}{|\underline{r} - \underline{r}'|} dv' \quad (8)$$

In engineering applications j is often given by the circuit geometry and A is readily calculated (often, B is directly derived in such cases from the Biot-Savart formula). In space, on the other hand, j is generally a derived quantity obtained only after B is given, often with low accuracy, so that (8) is used relatively infrequently.

One important case in space science where j is given occurs in regions where it is inferred from the observed fluxes of charged particles

[e.g. Northrop, 1963, eq. 4.1 ; Longmire, 1963] . One such region occurs in the vicinity of the earth, where trapped particles support a current density often called (not entirely accurately) the ring current. The magnetic field in these regions is generally known from independent observations and its comparison to \underline{j} forms an interesting check, often enabling one to deduce which part of an observed particle population contributes most to the magnetic perturbation. Many analyses of the ring current obtain \underline{B} directly from \underline{j} via the Biot-Savart formula [e.g. Akasofu and Chapman, 1961 ; Schild, 1969] . However, general formulas for \underline{A} , derived by (8) for configurations used in calculations of the effects of a ring current, have been published by Kendall et al. [1966] .

Another case in which distributions of current density are used as the source of magnetic fields in space occurs when such distributions are introduced as the input data of theoretical models. The configuration of the current in such cases may be selected on physical grounds - e.g. field-aligned currents [Bonnevier et al., 1970 ; Crooker and Siscoe, 1974] or the geomagnetic tail sheet current [Williams and Mead, 1965] - or else it (or part of it) may be represented by a generally expanded function of position. The advantage of using \underline{j} in the latter case is that this assures the vanishing of $\nabla \cdot \underline{B}$, which may have been the reasoning of Olson [1974 ; Olson and Pfitzer, 1974] . Later on in this review, however, it will be shown that similar properties may be obtained by simpler means. In most such applications, the Biot-Savart formula is used and \underline{B} is derived directly.

(2) THE SCALAR POTENTIAL γ

(a) INTRODUCTION

In current-free regions $\nabla \times \underline{B}$ vanishes and it is possible to represent \underline{B} by a scalar potential

$$\underline{B} = - \nabla \gamma \quad (9)$$

Because $\nabla \cdot \underline{B} = 0$, γ is harmonic

$$\nabla^2 \gamma = 0 \quad (10)$$

and may be expanded in a variety of ways appropriate for harmonic functions. The properties of harmonic functions and of γ in particular are discussed in most texts of classical electrodynamics and for this reason the discussion here will be confined to applications relevant to the geomagnetic field.

(b) SPHERICAL HARMONICS

The "main" geomagnetic field - i.e. that part of the field observed at or above the earth's surface which is caused by currents in the earth's interior - accounts for about 99 % of the field observed at ground level and is readily expanded in external spherical harmonic functions

$$\gamma = a \sum_{n=1}^{\infty} \sum_{m=0}^{m=n} (a/r)^{n+1} P_n^m(\theta) (g_n^m \cos m\varphi + h_n^m \sin m\varphi) \quad (11)$$

where a is the earth's radius and P_n^m are associated Legendre functions. Attention should be given to the factor a outside the summation, intended to give the coefficients g_n^m and h_n^m the dimensions of magnetic

field : as a consequence such coefficients are generally given in units of gauss or gammas (one gamma = $1 \gamma = 10^{-5}$ gauss). The form given in (11), involving real coefficients, is preferred in geomagnetic research over the one used in mathematical treatments, where the last factor in each term of (11) is expressed in terms of $\psi_n^m \exp(im\varphi)$, with the summation over m extending from $-n$ to n and the coefficients (ψ_n^m, ψ_n^{-m}) being complex conjugates (a related representation uses spherical harmonic functions $Y_n^m(\theta, \varphi)$ proportional to $P_n^m(\theta) \exp im\varphi$). Properties and details of this expansion are reviewed in many texts and articles [e.g. Chapman and Bartels, 1940 ; Heppner, 1963 ; Kaula, 1968 ; Stacey, 1969 ; Zmuda, 1973] .

In using the spherical harmonic expansion of γ note should be taken of the choice of normalization, for several definitions of Legendre functions, differing by constant factors, are currently employed.

Three main choices of this kind are described by Chapman and Bartels [1940] ; in the notation used by them (adopted from Adolf Schmidt) Legendre functions are denoted $P_{n,m}$ for mathematical normalization, $P^{n,m}$ for gaussian normalization and P_n^m for Schmidt normalization. The same notation will be used here but it is by no means a general one and frequently authors denote Legendre functions by P_n^m regardless of the normalization which they are using.

In the above notation, if $\mu = \cos\theta$

$$P_{n,0}(\mu) = \left((-1)^n / 2^n n! \right) d^n / d\mu^n (1 - \mu^2)^n$$

(Legendre polynomials) (12)

$$P_{n,m}(\mu) = (1 - \mu^2)^{m/2} d^m P_{n,0} / d\mu^m$$

(13)

$$P^{n,m}(\mu) = \left[(n-m)! / 1.3. \dots (2n-1) \right] P_{n,m}(\mu)$$

$$= \left[(n-m)! n! 2^n / (2n)! \right] P_{n,m}(\mu)$$

(14)

$$P_n^0 = P_{n,0} \quad (15)$$

$$P_n^m = \left[2 (n - m)! / (n + m)! \right]^{1/2} P_{n,m} \quad (m \neq 0)$$

$$P_n^0 = \left[1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 1) / n! \right] P_n^{n,0} \quad (16)$$

$$P_n^m = \left\{ 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 1) / \left[\frac{1}{2} (n + m)! (n - m)! \right] \right\}^{1/2} P_n^m$$

(m ≠ 0)

The coefficients of (11) generally have their indices arrayed in the same manner as the Legendre functions to which they are attached. In transforming (11) from one normalization to another, whenever a Legendre function is multiplied by some factor, the corresponding harmonic coefficient has to be divided by the same factor, ensuring that γ does not vary.

In the literature coefficients of the expansion of γ are generally listed for either Schmidt or gaussian normalization: for easy identification the convention exists to reverse all coefficient signs for gaussian normalization, so that the axial dipole coefficient g_1^0 is negative for Schmidt-normalized potentials but positive for gaussian ones. In Schmidt normalization the magnitude of terms is roughly of the order of their contribution to the field: their gradual decrease with growing n reflects the relative preponderance (near the earth's surface) of the contribution of low-order harmonics, and for any n they do not vary systematically with m .

In gaussian normalization the magnitude of terms decreases more gradually with n and for any n the terms having low m tend to be larger. In computer applications the derivation of $P_{n,m}$ is, however,

much simpler than that of P_n^m , because it can be handled by means of a recursion relation free from any irrational factors, connecting terms with the same m [Cain et al., 1967] :

$$P_{m,m}(\theta) = \sin^m \theta$$

$$P_{m+1,m}(\theta) = \cos \theta \sin^m \theta \quad (17)$$

$$P_{n+1,m}(\theta) = \cos \theta P_{n,m} + \frac{(n+m)(n-m)}{(2n+1)(2n-1)} P_{n-1,m}$$

The derivatives required for the calculation of B_θ are best found by a recursion relation based on the derivatives of the above relations, starting with

$$dP_{m,m}/d\theta = m \cos \theta \sin^{m-1} \theta \quad (18)$$

Normalizations other than those described also exist: a comparative list of 8 choices from 18 choices has been compiled by Kaula [1965 ; Table 1] .

(c) GEOMAGNETIC MODELS

Expansions of the form (11) have been used for the geomagnetic field since the time of Gauss [1839] . Typically, model expansions use harmonics with n ranging up to n_{\max} , with n_{\max} between 6 and 10 . Since the number of independent coefficients in such an expansion is $(n_{\max} + 1)^2 - 1$, one often finds authors speaking about models with 48, 63, 80, 99 or 120 coefficients.

A very comprehensive review of early analyses of the main field has been compiled by McDonald and Gunst [1967] and some early results are also given by Chapman and Bartels [1940]. More recent field models have been listed by Heppner [1963] (gaussian normalization) and by Vestine [1960] (Schmidt-normalized). Cain [1971] reviewed some other recent work and analyzed problems which arise in connection with field mapping by satellite. An International Geomagnetic Reference Field IGRF 1965.0 was derived and published by IAGA Commission 2, Working Group 4 [1969] and contains 80 terms.

Most of these models take into account the slow "secular" variation of the field by assuming a linear dependence of the coefficients (g_n^m, h_n^m) of the form

$$a = a_0 + \dot{a} t \quad (19)$$

where a_0 is the value of the coefficient at some initial time (e.g. the beginning of 1965 for the IGRF model mentioned before) and t is in years. Some models - for example that of Cain et al. [1967] - also include correction terms proportional to t^2 .

(d) CURRENT-FREE MODELS OF THE MAGNETOSPHERE

The expansion (11) consists solely of "internal harmonics" representing a field which originates inside the earth and vanishes at infinity. If sources external to the earth were also included their contribution to γ could be represented by a series of "external harmonics" with positive powers of r

$$\gamma' = a \sum_{n=1}^{\infty} \sum_{m=0}^{m=n} (r/a)^n P_n^m(\theta) [\bar{g}_n^m \cos m\varphi + \bar{h}_n^m \sin m\varphi] \quad (30)$$

Models of this form have been proposed for the magnetospheric field by Mead [1964] and by Midgeley [1964]. In both cases the earth's

field was approximated by the axial dipole given by g_1^0 in (11), which is orthogonal to the earth-sun line, and the frame of coordinates for both the dipole component and the expansion (20) is fixed with respect to the sun and the plane of the ecliptic.

In more complicated models internal and external fields are usually expressed in different frames of reference, since the internal field alone co-rotates with the earth. If one neglects all internal harmonics except for the three dipole coefficients g_1^0 , g_1^1 and h_1^1 , the model will depend (when external conditions are fixed) only on one parameter which varies in time - the angle between the earth's dipole axis and the earth-sun line, which varies by about $\pm 35^\circ$ around 90° , depending on the season of the year and the rotation of the earth. For any given value of this angle such a model is symmetrical about the plane containing the dipole axis and the earth-sun line and it is conveniently described in solar geomagnetic coordinates [Olson, 1969 (where the caption of Fig. 1 should be ignored); Russell, 1971; Burch and Janetzke, 1974] with the origin at the earth's center, the x-axis pointing sunward, the x-z plane containing the dipole axis and the z-axis pointing into the northern hemisphere.

Current-free models of this kind are not meant to represent external fields observed on the earth's surface - indeed, the surface effects predicted by such models are far too small to account for the observed daily variation [Mead, 1964]. Instead, they are intended to approximate the large-scale field of the outer magnetosphere. In a qualitative way they seem to agree with observations - in particular, they display a sharp boundary on the sunward side, limiting the region of field lines connected to the earth and corresponding to the observed magnetopause. The most significant non-dipole coefficients seem to be \bar{g}_1^0 and \bar{g}_2^1 and further improvement has been obtained [Williams and Mead, 1965] by adding a current sheet across the median plane of the geomagnetic tail.

Another simple current-free model approximating the external magnetosphere is the image dipole model; here the effects of the external field are approximated by a dipole parallel to the z-axis but located some distance sunward from the earth [Hones, 1963; Taylor and Hones, 1965; Forbes and Speiser, 1971] . This representation, which was inspired by the theory of the magnetic storm developed by Chapman and Ferraro [1931] , has two adjustable parameters - the distance to the "image dipole" and its magnetic moment, which generally exceeds that of the earth by a considerable factor. For instance, in the work of Taylor and Hones [1965] the image dipole is 28 times stronger than the axial dipole of the earth (other internal terms are ignored) and is placed 40 earth radii sunward of the earth, at a point which is outside the "magnetopause" and thus beyond the region in which the model is valid; the model also includes a sheet current in the geomagnetic tail region, somewhat similar to the sheet current introduced by Williams and Mead [1965] . In all models with image dipoles the scalar and vector potentials are readily found by superimposing the contribution of the two dipoles, and it is usually best to leave them in this form and not expand γ in spherical harmonics.

(e) CHANGE OF COORDINATES

In general the expansion (11) is given in spherical coordinates with the origin at the earth's center and the z-axis aligned with the terrestrial rotation axis. For some applications it is useful to rotate the z axis so that it coincides with the dipole axis, giving the so-called "dipole coordinates" [Chapman, 1963] . If γ is expanded in dipole coordinates both g_1^1 and h_1^1 vanish and the entire dipole contribution is contained in the axial dipole term g_1^0 .

The use of dipole coordinates is especially advisable when it is desired to enhance the preponderance of the axial dipole component over other harmonic terms. For instance, geomagnetic Euler potentials (described later) may be obtained by a perturbation calculation in which

the zero-order term is the contribution of the dipole field; it is then clearly advantageous to use dipole coordinates which make this term relatively large and reduce the remaining "perturbation".

Mathematical formulas exist for transforming (11) from geographic to dipole coordinates and they have the virtue that harmonics of a given lower index n contribute in the new coordinates only to harmonic terms with the same n . Thus each group of $2n+1$ coefficients corresponding to a given n transforms independently and (say) a model with 99 coefficients ($n \leq 9$) will be represented with full accuracy by the same number of terms in the new coordinates (to be precise, there will only be 97 independent terms in the new expansion, since g_1^1 and h_1^1 vanish). Formulas for such a transformation have been discussed by James [1969] and are concisely given by Slater [1960] .

Where high-speed computers are available it is often simpler to apply a "brute force" transformation as follows. Suppose a 99-term expansion of γ is given: one then begins by deriving its value from the given expansion (in geographical coordinates) at 97 points scattered over the surface of the earth. Next one transforms the coordinates of each of the selected points to (spherical) dipole coordinates in which the z axis is antiparallel to the vector having cartesian components (h_1^1, g_1^1, g_1^0). In the new coordinates the expansion of γ at each of the given points is derived, with the new expansion coefficients (g_n^m, h_n^m) entering as 97 unknown quantities (two of them vanish and are not counted). This produces 97 equations in 97 unknowns, which are now solved to give the new coefficients. A repetition based on another set of points is recommended as a check on accuracy and as a precaution against ill-conditioned sets of equations, but the resulting set of coefficients is usually accurate enough for normal use. An example of this method is given by Stern [1971] who included in the transformation not only the time derivatives of (19) but also the slow secular variation of the new coordinate axes.

If the origin of the coordinates is also allowed to vary we obtain eccentric dipole (or "offset dipole") coordinates. The choice of an arbitrary origin introduces 3 more adjustable parameters and can be used, for instance, to assure the vanishing not only of g_1^1 and h_1^1 but of g_2^0 , g_2^1 and h_2^1 as well [Bartels, 1936; Chapman and Bartels, 1940]. The z axis of eccentric dipole coordinates should parallel that of dipole coordinates to assure the vanishing of the two off-axis terms with $n = 1$.

Unfortunately, a given expansion in geographic or dipole coordinates (e.g. with 99 coefficients) is no longer accurately represented by the same number of coefficients (or in general, by any finite number) once the origin is shifted. In case of the 80-term IGRF 1965.0 model, Hilton and Schulz [1973] have shown that a 195-term representation ($n \leq 13$) of the field in offset dipole coordinates maintains sufficient accuracy for practical purposes and have published its coefficients. It may also be noted here that coordinates of the type discussed here seem to be useful in describing Jupiter's main magnetic field, as observed by the Pioneer 10 spacecraft [Smith et al., 1974].

(f) PRACTICAL PROBLEMS RELATED TO UNIQUENESS

The practical derivation of γ for the main geomagnetic field involves many observational problems, such as the elimination from the data of effects of external sources and of magnetic anomalies in the earth's crust. Assuming these to be solved, the problem can be viewed as involving the derivation of

$$\gamma = \gamma_0 + \gamma_1 \quad (21)$$

where γ_0 is some previously known model of the potential and γ_1 is a relatively small correction which is to be derived from the given data. If γ_1 is expanded as in (11) in spherical harmonics involving some number N of unknown coefficients, then any observation related to the

field can be reduced to a linear equation involving these coefficients. The relation may be exact, e.g. when any component of \underline{B} is observed, or it may be a linearized approximation, as happens when one observes the field's magnitude $|\underline{B}|$ (see later on) or any of the direction angles defining \underline{B} .

Thus the results of a world-wide magnetic survey tend to reduce to a large set of linear equations in N unknowns. The coefficients of γ_1 are then derived by the least squares procedure as those coming closest to fitting the set and, in case approximate linearization was used in obtaining the equations, γ_1 is generally added to γ_0 to give an improved initial approximation, after which the procedure is repeated one or more times [Cain et al., 1967].

An interesting problem related to such procedures is that of the uniqueness of the result: how can one be sure that a given data set leads to a unique choice of γ ?

One case in which this question has led to unexpected results involves the derivation of γ from observations of $|\underline{B}|$ (commonly denoted in geomagnetism by F , a practice which will be followed here). Such observations are easily performed aboard spacecraft with an accuracy of about one gamma [Cain, 1971 and references cited there] and they have the advantage of not requiring precise knowledge of the attitude of the sensor. If one neglects the altitude variation of the orbit, the uniqueness problem reduces to the question whether γ is uniquely determined by observation of $|\nabla \gamma| = F$ over the surface of a sphere.

Linearizing the expression of F obtained from (21) gives

$$\nabla \gamma_0 \cdot \nabla \gamma_1 = \frac{1}{2} [F^2 - (\nabla \gamma_0)^2] \quad (22)$$

and this can be used iteratively (as described before) to derive the coefficients of γ , with γ_1 added to γ_0 after each iteration step to provide a better starting approximation for the step following. In

computer simulations in which a known expansion (11) was used to generate F this method recovered the input coefficients quite accurately within 4-5 steps. Consequently it was generally accepted and widely used in the treatment of actual data.

Meanwhile, however, Backus [1968, 1970, 1974] investigated the problem mathematically. He first found that F and γ were uniquely related whenever the series (11) was finite [Backus, 1968], then he showed by actual counterexamples that this was no longer universally true if the series (11) was infinite [Backus, 1970] and finally he proved that this ambiguity was removed if F was observed over a finite volume in space [Backus, 1974].

At the same time evidence began to accumulate suggesting that models derived from F observed in near-circular orbits fit the vector field far less accurately than they fit the distribution of the field's magnitude. In addition, different models derived from F exhibited relatively large differences and this prompted Hurwitz and Knapp [1974] to conduct simulated recoveries similar to those described before but with data contaminated by finite "noise", as occurs in practical situations. In such cases γ is not recovered exactly but finite errors remain and Hurwitz and Knapp found that the fit between the input vector field and that derived from the output model was decidedly inferior to the fit between input F and output F .

Stern and Bredekamp [1975] independently obtained similar results and also showed that such effects were connected to the counterexamples of Backus [1970]. These counterexamples bear a special relation to the dipole field and the fact that the main geomagnetic field is dominated by its dipole component establishes a connection between them and the problem. In particular, it turns out that in the presence of finite "noise" certain sequences of harmonic terms can exhibit enhanced fluctuations which degrade the fit to vector data much more than they degrade the fit to F . Such enhanced fluctuations were in fact obtained in

computer simulations and also in analysis of some of the runs of Hurwitz and Knapp [1974] and this strengthens the suspicion that some current models based on F are in fact less accurate than has been claimed.

The uniqueness of γ derived from some other types of data has been examined by Kono [197] in order to evaluate the correctness of some reconstructions of ancient geomagnetic fields. For instance, he proved that magnetic declination observed at the earth's surface does not in general define γ uniquely.

(3) TOROIDAL AND POLOIDAL VECTORS

(a) INTRODUCTION

A general vector field \underline{V} may be represented by three scalars Ψ_0 , Ψ_1 and Ψ_2 , in the form

$$\underline{V} = \nabla \Psi_0 + \nabla \times \Psi_1 \underline{r} + \nabla \times \nabla \times \Psi_2 \underline{r} \quad (23)$$

The advantage of this form is (as can be verified by carrying out the algebra) that if \underline{V} satisfies the vector Helmholtz equation

$$\nabla^2 \underline{V} + k^2 \underline{V} = 0 \quad (24)$$

then each of the Ψ_i satisfies the appropriate scalar Helmholtz equation

$$\nabla^2 \Psi_i + k^2 \Psi_i = 0 \quad (25)$$

(including the case $k^2 = 0$, when (25) becomes Laplace's equation). In a similar way, if equation (24) is written in cartesian coordinates, it is resolved into three scalar equations of form (25), one for each cartesian

component of \underline{V} ; this is useful if the boundary conditions are easily expressed in cartesian coordinates, whereas (23) bears a similar relation to spherical coordinates. Only a few systems of coordinates allow such direct conversion of the vector equation to the corresponding scalar one [Senior, 1960; Morse and Feshbach, 1953, chapt. 13] .

Because (24) arises naturally in wave propagation problems, the representation (23) was first introduced in that context [Mie, 1908 ; Hansen, 1935]. If a magnetic field \underline{B} is represented in this manner, Ψ_0 is generally omitted, since it must then be a harmonic function and its contribution (as will be shown) may be represented by part of Ψ_2 . The representation using this form

$$\underline{B} = \nabla \times \Psi_1 \underline{r} + \nabla \times \nabla \times \Psi_2 \underline{r} \quad (26)$$

was first introduced in connection with geomagnetic dynamo theory by Elsasser [1945, 1946, 1947, 1956] who called the two components the toroidal and poloidal components of \underline{B} , respectively.

These names are still used. To get some intuitive feeling for their significance, consider axisymmetrical fields, in which neither Ψ_1 nor Ψ_2 depends on the azimuth angle φ . A toroidal field of this type is then aligned with the φ direction and has field lines circling the z axis, while the field lines of a poloidal axisymmetric field are orthogonal to the φ direction and are confined to meridional planes. as another axisymmetrical example, consider a poloidal field with field lines covering the surfaces of a family of nested toruses. When a toroidal component is added to this field, field lines acquire an azimuthal slant so that instead of staying in planes of constant φ they spiral around the axis of their torus, in a way resembling that found in toroidal plasma confinement devices.

Some of the elementary applications of (26) to dynamo theory have been described by Elsasser [1956] . For instance, in a conducting sphere both components of (26) will tend to decay with time: differential

rotation - such as is observed on the sun and Jupiter, both of which have magnetic fields - will amplify the toroidal component, but the strength of this process is proportional to the poloidal field which ultimately decays, unless it is continuously maintained by some feedback mechanism originating in the toroidal component. Parker [1955] showed that radial flow in the rotating frame may induce cyclonic swirling which indeed leads to such feedback. Various theories of solar magnetism based on such considerations have been advanced [Babcock, 1961 ; Leighton, 1969] but the details are beyond the scope of this review.

(b) VECTOR POTENTIAL FOR A CURL-FREE FIELD

Two identities useful in handling (26) are

$$\nabla \times \psi_1 \underline{r} \equiv \nabla \psi_1 \times \underline{r} \quad (27)$$

$$\nabla \times \nabla \times \psi_2 \underline{r} \equiv \nabla \partial/\partial r (r \psi_2) - \underline{r} \nabla^2 \psi_2 \quad (28)$$

If B is curl-free and satisfies (2), with γ expanded as in (11) and (20), then it can be represented as a poloidal field with ψ_2 a harmonic function simply related to γ [Stern, 1964]. To see that, note that if ψ_2 is a series of spherical harmonics, the same holds true for $\partial/\partial r (r \psi_2)$. Furthermore, the last term of (28) vanishes in that case, so that only the gradient of a harmonic function remains. This allows formal identification

$$\gamma = - \partial/\partial r (r \psi_2) \quad (29)$$

and a term-by-term comparison of expansion coefficients. Specifically, if γ has expansion coefficients $(g_n^m, h_n^m, \bar{g}_n^m, \bar{h}_n^m)$ as in (11) and (20) and ψ_2 has corresponding coefficients $(G_n^m, H_n^m, \bar{G}_n^m, \bar{H}_n^m)$, then

$$\begin{aligned} G_n^m &= g_n^m/n & H_n^m &= h_n^m/n \\ \bar{G}_n^m &= -g_n^m/(n+1) & \bar{H}_n^m &= -h_n^m/(n+1) \end{aligned} \quad (30)$$

To obtain the vector potential \underline{A} for a given γ , expanded as in (11) and (20), equations (30) are used to derive the expansion terms of Ψ_2 , after which one calculates

$$\underline{A} = \nabla \times \Psi_2 \underline{r} \quad (31)$$

(c) MODELS OF THE MAGNETOSPHERE

Curl-free models of the far-away geomagnetic field, as were described in the section on the scalar potential, are incapable of great quantitative accuracy, because the region which they describe contains an appreciable current density. As an alternative, Mead and Fairfield [1975; Fairfield and Mead, 1975] tried to represent the field in such regions by expanding each component of \underline{B} in powers of cartesian coordinates

$$B_i = \sum_{n,r,s} a_{inrs} x^r y^s z^{n-r-s} \quad (32)$$

Potential fields such as those of (20) are easily reduced to this form but they comprise only a limited subclass of such fields.

Two problems, arise, however: the first is how to assure the vanishing of $\nabla \cdot \underline{B}$, and the second is the difficulty in controlling the model near the boundary of the region for which it is derived, due to the presence of positive powers of x , y and z .

To overcome the first difficulty Mead and Fairfield derived the relations between the coefficients of (32) which are required in

order for $\nabla \cdot \underline{B}$ to vanish. They then analyzed magnetometer data from space and derived a "best fitting set" of coefficients for (32) by a least-squares procedure similar to the one used for deriving the main field, described in section (2-f), but based on vector data. In that calculation the relations which assure the vanishing of $\nabla \cdot \underline{B}$ were introduced as constraints and were handled by the method of Lagrange's multipliers.

A simpler method is however available: noting that

$$\underline{r} = \hat{x} x + \hat{y} y + \hat{z} z \quad (33)$$

and expanding (26) in cartesian coordinates shows that if ψ_1 and ψ_2 are expanded in the form (32), then the resulting components B_i also have this form. In particular, if the largest power n of variables having the dimension of length is N for ψ_1 and $N-1$ for ψ_2 , then its value for the expansion of B_i will be N .

Not all terms in the expansions of ψ_i produce independent contributions to \underline{B} , because any part of ψ_i which depends on r alone does not affect \underline{B} . The terms with $n = 0$ belong to this class and should therefore be omitted. Consider next the three quadratic terms proportional to x^2 , y^2 and z^2 . These can be combined to give 3 independent terms proportional to x^2 , y^2 and r^2 and the last of these does not contribute to \underline{B} , so that it can be omitted. Similar arguments show that for any integer k , one coefficient can be eliminated among those representing homogeneous polynomials of degree $2k$ in the expansion of ψ_1 or ψ_2 .

One advantage of this approach is that no precautions are required to ensure the vanishing of $\nabla \cdot \underline{B}$. It is also easily generalized to overcome the second difficulty mentioned earlier, by devising models in which ψ_1 and ψ_2 are expanded in the form

$$\psi_j = \sum_{n, m, s} a_{jnms} x^m y^s z^{n-m-s} \exp(-r/r_0) \quad (34)$$

where r_0 is some chosen scale distance (in principle, several series with different choices of r_0 could be used). It is more convenient to use in this case spherical harmonics

$$\psi_j = \sum_{n, m, s} r^n \exp(-r/r_0) P_m^s(\theta) \left[u_{jnms} \cos s\varphi + v_{jnms} \sin s\varphi \right] \quad (35)$$

from which the components of \underline{B} are readily derived. It should be noted that the expansion of (34) is not equivalent to that of (35); in fact, the latter expansion is equivalent to

$$\psi_j = \sum_n r^n \sum_{u,v,w} a_{jnuvw} (x/r)^u (y/r)^v (z/r)^w \exp(-r/r_0)$$

which includes the terms of (34) if $u+v+w = n$ but also contains additional expressions. Some comparisons between this method and that of Mead and Fairfield have been performed (D. Stern, unpublished); while the improvement is rather small compared to the inherent dispersion of the observational data, this approach should be useful in future studies by providing more flexible analytical expansions.

Magnetospheric models somewhat similar to those described here have been used by Olson and Pfitzer [1974]. However, the derivation of these models does not involve ψ_i : it starts by fitting a system of currents to observed fields [Olson, 1974] and then approximates the cartesian components of their fields by general expansions similar to (32) and (34). The resulting models are not automatically divergence-free.

It should be stressed that there exists no assurance that "global" models, representing the entire magnetosphere by one series of analytic functions of a given form, can do so with arbitrary accuracy. The power series (32), for instance, is an ordinary Taylor expansion and can represent functions only in a certain neighborhood of the origin, where it converges: this region may not only be finite, it may be smaller than the magnetosphere, in which case any representation of the field in this way contains a finite error. The representation (34) appears at first sight to be more useful but it, too, is subject to the same restrictions since it just represents the cartesian expansion of $\psi_i \exp(r/r_0)$.

At present, observational inaccuracies are of the same order as those introduced by global models and such models are therefore in general use. Ultimately, however, it may be necessary to adopt a different approach. Two extreme cases could then be considered - an extension of the global method by which the magnetosphere is divided into a number of large regions, each with its own expansion, and numerical representation by means of a tabulated network of values (e.g. of ψ_i). It should be realized, however, that the numerical representation can also be viewed as a local expansion, by means of the interpolation formulas used, and by making the numerical grid more sparse and the interpolation formula more powerful, the gap between these two extremes could in principle be bridged.

(a) SPHERICAL VECTOR HARMONICS

The spherical harmonic functions $P_n^m(\theta) \cos m\varphi$ and $P_n^m(\theta) \sin m\varphi$ - or, alternatively

$$X_n^m = P_n^m(\theta) e^{im\varphi} \quad (36)$$

(in the notation of Morse and Feshbach [1953], p. 1898-1900, which this section will adopt) - form a complete set in which arbitrary analytic functions of (θ, φ) can be expanded. Similarly, spherical vector harmonics are 3 sets of vector functions of θ and φ , denoted as $\underline{P}_{n,m}$, $\underline{B}_{n,m}$ and $\underline{C}_{n,m}$, which find use in the expansion of vector functions.

Many problems in wave propagation lead to a vector field \underline{V} represented as in (23) and satisfying (24) for some values of k . This gives 3 independent equations of the form (25) and separation of the r variable shows then that ψ_i have the form

$$\psi_i = \sum_{n,m} A_{nmi} F_{nm}(r) X_n^m(\theta, \varphi) \quad (37)$$

where $F_{nm}(r)$ stands for Hankel functions if $k \neq 0$ and powers of r if $k = 0$. When (37) is then substituted in (23) it is found that all terms are expressed as the product of functions of r with some member of one of the following three families of spherical vector harmonics

$$\begin{aligned} \underline{P}_{n,m} &= \underline{r} X_n^m(\theta, \varphi) \\ \underline{B}_{n,m} &= r [n(n+1)]^{-1/2} \nabla X_n^m(\theta, \varphi) \\ \underline{C}_{n,m} &= [n(n+1)]^{-1/2} \nabla \times \underline{r} X_n^m(\theta, \varphi) \end{aligned} \quad (38)$$

Note that any dimensional dependence on r in the above definitions cancels out and also, by (27), that for any pair of values of the indices n and m the three above vectors are orthogonal. Other features of these vectors (and of their real and imaginary parts, since by (36) they represent complex quantities) are described by Morse and Feshbach [1953].

Many of the representation discussed earlier can be easily expressed in terms of spherical vector harmonics. For instance, the expansion (35) is formally the same as that of (37) and therefore the results of its substitution in (26) can be expressed in terms of the vectors listed in (38).

(4) EULER POTENTIALS

(a) DEFINITION AND GENERAL PROPERTIES

An intuitive description of a magnetic field, first introduced by Michael Faraday, is provided by magnetic field lines (or "lines of force" as he termed them) - lines which are everywhere tangential to \underline{B} . Such a description is completely analogous to the representation of the velocity field of an incompressible fluid by means of streamlines.

For a long time field lines were mainly used as a device for visualizing the magnetic field. However, with increasing interest in particle motion and transport properties in a near-collisionless plasma - both in space and in the laboratory - field lines assumed new importance, since both these effects are channeled along them. This created interest in mathematical descriptions of the magnetic field which explicitly contain representation by field lines; such a description has been available in fluid dynamics and it involves two scalar functions α and β such that

$$\underline{B} = \nabla\alpha \times \nabla\beta \quad (39)$$

The functions (α, β) are generally known as Euler potentials because it was Leonhard Euler who first introduced such a representation in the context of fluid dynamics; they appear to have been introduced into plasma physics by Grad [Northrop and Teller, 1960; Gardner, 1959]. A short review of their history and properties was given by Stern [1970] and the reader is referred to that work for details extending beyond the present brief description.

The basic properties of (α, β) are easily derived from (39). We have

$$\begin{aligned} \underline{B} \cdot \nabla\alpha &= 0 \\ \underline{B} \cdot \nabla\beta &= 0 \end{aligned} \quad (40)$$

This means that surfaces of constant α and β are everywhere tangential to \underline{B} - and this property extends to lines along which two of such surfaces intersect, which thus are field lines. Note that (39) implies (40), but not vice versa. Two functions (u, v) with the property of (40)

$$\underline{B} \cdot \nabla u = \underline{B} \cdot \nabla v = 0$$

do not satisfy (39), in the general case, but rather

$$\underline{B} = w (\nabla u \times \nabla v)$$

where w is an arbitrary function of (u, v) . Functions such as (u, v) may be called unmatched Euler potentials and have been introduced by Sweet [1950] and Dungey [1958]. They are conserved along field lines and are therefore functions of (α, β) .

In general (subject to restrictions of uniqueness and single valuedness noted later) field lines of a given configuration form a two-parameter family, in which the field line corresponding to the values (α_0, β_0) of the parameters are represented by the intersection of the surfaces

$$\begin{aligned} \alpha(x, y, z) &= \alpha_0 \\ \beta(x, y, z) &= \beta_0 \end{aligned} \tag{41}$$

As an example, a simple model of the geomagnetic field is given by an axial dipole of moment g_1^0 ; if (as in eq. 11) a represents the earth's radius, one convenient choice of (α, β) , in spherical coordinates (r, θ, φ) , is given by

$$\alpha = a g_1^0 (a/r) \sin^2 \theta \tag{42-a}$$

$$\beta = a \varphi \tag{42-b}$$

Given any two numerical values (α_0, β_0) of α and β , a field line of this field is defined by (41). Note that (42-b) indicates that field lines lie in constant meridional planes, while (42-a) gives the shape of a field line within such a plane. Other examples will be presented later.

Euler potentials are not uniquely defined: for instance, α can be incremented by any function of β without (39) being violated. More generally, (α, β) may be replaced by (α', β') , provided the new Euler potentials are functions of the old ones and the Jacobian of the transformation is unity:

$$\partial(\alpha', \beta') / \partial(\alpha, \beta) = 1 \quad (43)$$

Given a magnetic field represented as in (40), it is generally possible to choose for α' (at least inside a restricted region in space) any well-behaved function $\alpha'(\alpha, \beta)$ and then derive a corresponding conjugate Euler potential β' . An application of this property will be presented at the end of this section.

A vector potential corresponding to (39) is

$$\underline{A} = \alpha \nabla \beta \quad (44)$$

and it has the property $\underline{A} \cdot \underline{B} = 0$.

With all these useful properties, there exist two important drawbacks to Euler potentials which limit their application. The first is non-linearity: the representation (39) is non-linear, since it involves products of the derivatives of α and β . This means that superposition does not hold for Euler potentials: given the Euler potentials for two fields \underline{B}_1 and \underline{B}_2 , the functions (α, β) for the combined field does not equal the sum of the Euler potentials expressing \underline{B}_1 and \underline{B}_2 separately and, indeed, might be difficult to derive, even when the Euler potentials of the component fields are known. In practice this greatly limits the class of fields for which analytic forms or even analytic approximations of (α, β) are readily derived.

The other drawback, more important in the laboratory than in space applications, is the possibility that the labeling of field lines by (α, β) is not single valued. In toroidal confinement devices, for instance (tokamaks, stellarators), if a field line is labeled by (α, β) in some limited region of the field and is then followed outside this region, it may ultimately enter the region once more. In fact, such a field line usually returns to the same region again and again and with each return it coincides with some previously labeled field line - in general, one with different values of (α, β) . In such cases the labeling is single valued only if a limited region is considered and is not possible when the entire field is represented. For further details about these and other properties of Euler potentials the reader is referred to the review by Stern [1970] .

As an illustration of an application consider the motion of trapped particles in the magnetosphere under the combined influence of a static electric field

$$\underline{E} = - \nabla \phi$$

and the geomagnetic field \underline{B} . Because of the high conductivity along field lines the electric field (except in some special regions, perhaps) comes close to being orthogonal to \underline{B} , i.e.

$$\underline{B} \cdot \nabla \phi = 0 \quad (45)$$

By (40) this condition is neatly expressed by the requirement that ϕ depends only on (α, β)

$$\phi = \phi(\alpha, \beta) \quad (46)$$

A charged particle of very low energy, conserving the two lowest adiabatic invariants but with negligible magnetic drift, will move in this field with the drift velocity

$$\underline{v}_E = (\underline{B} \times \nabla \phi) / B^2 \quad (47)$$

From this $\underline{v}_E \cdot \nabla \phi = 0$ and therefore the particle stays on a single equipotential surface throughout its motion.

Now as stated earlier, any "well-behaved" function α' of (α, β) can be chosen to replace α and an appropriate "conjugate potential" β' can then be found for it. In particular, let the role of α' be played by $\phi(\alpha, \beta)$ and let the conjugate potential be denoted $\psi(\alpha, \beta)$, i.e.

$$\underline{B} = \nabla \phi \times \nabla \psi \quad (48)$$

It is instructive to derive the rate at which ψ changes at the location of the drifting particle. By (47) and (48)

$$d\psi/dt = \underline{v}_E \cdot \nabla \psi = 1 \quad (49-a)$$

i.e.

$$\psi = t - t_0 \quad (49-b)$$

Thus a swarm of particles starting from a surface of constant ψ at $t = 0$ will always share the same value of ψ . The averaged motion of such particles - that is, the gradual change of their guiding field lines - is conveniently studied in the (α, β) plane, where to every pair of values of the Euler potentials - and hence, to every possible guiding field line - there corresponds one point (in the geomagnetic field the equatorial plane can play such a role). In this plane the lines

$$\phi(\alpha, \beta) = \text{constant}$$

can be viewed as rays along which very low energy particles propagate, while lines of constant ψ act as "wavefronts" marking the particles' progress. Figure (1), taken from Stern [1974] gives lines of constant ϕ (solid) and ψ (broken) for some particular geomagnetic model, in which the earth's field is approximated by a dipole field and some analytical model of ϕ is assumed.

The preceding is readily extended to particles of finite energy conserving the magnetic moment M and the second invariant J , except that now ϕ is replaced by the averaged guiding center hamiltonian $K(\alpha, \beta, M, J)$ introduced by Northrop and Teller 1960. This generalization has been developed for the dipole field by Chen and Stern [1975], who provided an analytical approximation for K and expressed its conjugate potential by means of numerical integration.

(b) EXAMPLES

In general, simple analytical forms for the Euler potentials can only be found for classes of fields with some type of symmetry. For instance, two-dimensional fields of the form

$$\underline{B} = B_x(x, y) \hat{x} + B_y(x, y) \hat{y} \quad (50)$$

have Euler potentials

$$\begin{aligned} \alpha &= \int B_x dy + f(x) = - \int B_y dx + g(y) \\ \beta &= z \end{aligned} \quad (51)$$

where the functions $f(x)$ and $g(y)$ are obtained by deriving α in two different ways (because $\nabla \cdot \underline{B} = 0$, two such equivalent derivations are possible). The hamiltonian for a particle with mass m and charge q moving in such a field can then be written

$$H = (1/2m) \left[p_x^2 + p_y^2 + (p_z - q \alpha(x, y))^2 \right] \quad (52)$$

Since p_z is a constant of the motion it follows that the motion reduces to that of a two-dimensional free particle in a non-negative potential

$$V = (p_z - q \alpha)^2 / 2m \quad (53)$$

For all such motions, lines of constant α in the (x, y) plane - which trace the pattern of field lines - are equipotentials of V . The value of V attached to any such line and the general configuration of regions of high and low potential depends on the constant p_z appropriate for the given motion. One interesting configuration of this class occurs in the neighborhood of an X-type neutral line [Russbridge, 1971]; other examples and generalizations have been examined by Stern [1975].

Axisymmetrical poloidal fields, by (26), can be represented by Euler potentials with $\beta = \psi$, since

$$\begin{aligned} \nabla \times \nabla \times \underline{r} \psi_2(r, \theta) &= - \nabla \times \hat{\phi} (\partial \psi_2 / \partial \theta) \\ &= - \nabla [r \sin \theta (\partial \psi_2 / \partial \theta)] \times \nabla \psi \end{aligned} \quad (54)$$

The dipole example of eqs. (42) belongs to this class. It is also possible to derive simple Euler potentials for pure toroidal fields:

$$\nabla \times \psi_1 \underline{r} = \nabla \psi_1 \times \nabla r \quad (55)$$

However (as noted earlier) when fields of both kinds are combined no simple way of deriving Euler potentials exists.

A case of practical interest involves the main geomagnetic field, usually represented by a scalar potential γ as in eq. (11). No analytical method is known by which (α, β) may be simply derived for such a field, but a perturbation technique is available for obtaining them approximately.

The method is based on the observed property that the dipole component g_1^0 in (11) dominates the expansion, especially if γ is expressed in dipole coordinates. In such coordinates let (α_0, β_0) be the dipole Euler potentials of eqs. (42) (the subscripts zero refer here to zero-order approximations and are unrelated to those appearing

in equations 41), while γ_0 is the corresponding scalar potential. Let $(\alpha_1, \beta_1, \gamma_1)$ be the first-order corrections, linear in the higher harmonic coefficients (g_n^m, h_n^m) , applied to these three quantities. Substitution in (39) then gives

$$-\nabla\gamma_1 = \nabla\alpha_0 \times \nabla\beta_1 + \nabla\alpha_1 \times \nabla\beta_0 \quad (56)$$

from which

$$\begin{aligned} \nabla\alpha_1 \cdot \nabla\gamma_0 &= \nabla\alpha_0 \cdot \nabla\gamma_1 \\ \nabla\beta_1 \cdot \nabla\gamma_0 &= \nabla\beta_0 \cdot \nabla\gamma_1 \end{aligned} \quad (57)$$

Expressing $(\alpha_1, \beta_1, \gamma_1)$ in terms of $(\alpha_0, \beta_0, \theta)$ allows these equations to be integrated and provides (α_1, β_1) within arbitrary functions of (α_0, β_0) . However, (57) contains only two of the three scalar relations implied by the vector equation (56); if the remaining scalar relation is now invoked, the arbitrary added function may be determined. Details have been derived by Stern [1967] and related expressions were first obtained in a different context by Pennington [1961, 1967]. The calculation has not been extended to higher order, except for one effort by Hassit [1967] in which second-order unmatched Euler potentials were derived.

The same perturbation method can also be applied to the external harmonics of section (2-3) and in particular to the simple model described there, with the expansion of γ involving only the three coefficients \bar{g}_1^0 , \bar{g}_1^{-1} and \bar{g}_1^2 . As might be expected, the resulting expansion breaks down at large distances, but it also becomes unrealistic near the z axis. It provides a fairly good representation of the field in the regions occupied by trapped particles; in the noon-midnight cross section of the (α, β) model in the region in which the model field departs markedly from the curl-free field from which it was derived, there exists the added feature (not found in the curl-free model) of a neutral point on the night side of the dipole, in addition to two such points on the day side, corresponding to the polar cusps. Further details have been given by Stern [1967] and Kosik [1971a, b].

A model with some similar properties (but much smaller deformation) has been developed by Parker [1960] .

While accurate representations of the magnetospheric field by means of (α, β) are difficult to obtain and no iterative procedures for deriving them have been published, it is relatively easy to devise simple models which exhibit appropriate qualitative properties. For instance, a distorted dipole field, compressed on one side and stretched out on the other, is given by

$$\begin{aligned} \alpha/a^2 g_1^0 &= \sin^2 \theta / r - k \sin^6 \theta (1 - \cos \varphi)^3 \\ \beta &= a \varphi \end{aligned} \quad (58)$$

where k is an adjustable constant. It is easy to distort this model further, in such a way that all its field lines are contained within the surface

$$f(r, \theta, \varphi) = 1 \quad (59)$$

This is accomplished if α is multiplied by $(1 - f)$; in order that the field will continue to approximate the dipole near the origin, f should tend to zero as $r \rightarrow 0$. For instance, if the field is to be contained inside the paraboloid

$$r = r_0 / (1 + \sin \theta \cos \varphi) \quad (60-a)$$

one can multiply α of (58) by the factor

$$1 - (r/r_0)^m (1 + \sin \theta \cos \varphi)^m \quad (60-b)$$

Increasing the adjustable parameter m narrows down the transition zone between the main field and the boundary field. Further improvement can be obtained by shifting the focus of the paraboloid away from the origin, but it should be noted that in all these models the boundary is always connected to the two dipole field lines which emerge along the z axis.

As a final note, axisymmetric models of this sort have been constructed by Barish et al. [1974] for the magnetic field of Jupiter, following its observation by Pioneer 10 [Smith et al., 1974] .

(c) THE L PARAMETER

Before the (α, β) system for the magnetosphere was developed an alternative approach to the labeling of magnetic field lines was introduced in the form of the so-called L parameter due to McIlwain [1961, 1966] .

As an intuitively meaningful quantity the L parameter is extremely useful: it has dimensions of length in units of earth radii and in a dipole field its value at any given point equals the maximum distance from the dipole attained by the field line passing the given point. In a perturbed dipole field (and the parameter is only defined for dipole fields and perturbed dipole fields) the value $L(r)$ at some given position r still approximately equals the largest distance from the origin which is attained by the field line through r . However, the generalization of L to perturbed dipole fields is not performed in a way which aims at preserving the equatorial crossing distance. Instead, the definition tries to preserve a different property, useful in studies of trapped radiation - that if a charged particle is trapped along any field line with a given value of L, as it gradually drifts around the dipole then all field lines which it passes will have the same value of L .

This property is only approximately achieved. Indeed, it cannot in general hold for all particle orbits in an asymmetric field, since two particles starting on a given field line in such a field with different mirroring points will in general follow slightly different surfaces ("drift shells" during their drift (this is known as shell splitting). However, for the geomagnetic field the separation between such surfaces tends to be small and consequently the values of L encountered by a particle in its drift generally vary by less than 2 % .

In this section the relation between L and (α, β) will be traced, the inherent limitations of L will be clarified and suggestions will be described for ways in which the concept can be generalized.

In a static near-dipole magnetic field, trapped particles tend to preserve the two lowest adiabatic invariants (the third invariant is not needed here)

$$\mu = p_{\perp}^2 / 2mB \qquad J = \oint p_{\parallel} dl \qquad (61)$$

where m is the rest mass, $(p_{\parallel}, p_{\perp})$ are the components of the momentum parallel and perpendicular to \underline{B} and integration is along a field line. Since the magnitude p of the momentum is also conserved (in the absence of electric fields, which will now be assumed) it is possible in this case to replace (μ, J) by two related quantities (B_m, I) which are also constants of the motion but do not depend on the particle's energy:

$$B_m = p^2 / 2m\mu \qquad (62)$$

$$I = J/2p = \int_{B_m}^{B_m} (1 - B/B_m)^{1/2} dl$$

where the integration of I extends between mirror points at which $B = B_m$. As the particle drifts from one guiding field line to the next, it always chooses that one of the adjacent field lines on which the value of I , evaluated between fixed values of B_m , is the same.

As the particle drifts its guiding field lines gradually trace a surface - a closed surface for trapped particles - called a drift shell or a drift surface. Because drift shells are tangential to \underline{B} their equations have the form

$$f(\alpha, \beta) = \text{constant} \qquad (63)$$

For each pair of parameters (I, B_m) there exists a drift shell and therefore the entire collection of drift shells can be represented by some function F as

$$F(\alpha, \beta, I, B_m) = \text{constant} \quad (64)$$

(the constant may be chosen as 0 or 1 without loss of generality).
Isolating α gives

$$\alpha = G(I, B_m, \beta) \quad (65)$$

In a dipole field β may be eliminated because of axial symmetry; in that case, if (α_0, β_0) are the Euler potentials defined in (42), then (65) reduces to

$$\alpha_0 = G_0(I, B_m) \quad (66)$$

Now (I, B_m) can be replaced as parameters characterizing drift shells by any two well-behaved functions depending on them [Stern, 1968]. In particular, in the case of the perturbed dipole field, $G_0(I, B_m)$ or any function $L(G_0)$ can be used as one such parameter, while B_m may be retained as the second one. This characterization will have the advantage that in the dipole limit - due to (66) - the shell equation depends only on a single parameter G_0 or $L(G_0)$, not on two. In that limit all the trajectories starting from the same initial field line trace the same surface and have identical values of G_0 or $L(G_0)$; by way of contrast, if such shells were classified by using (I, B_m) , a finite range in both these parameters would be required.

In a perturbed dipole field the shell equation is given by (66) and no single-parameter description is possible. However, if the perturbation is small, equation (65) will differ from (66) only by small correction terms. Denoting such terms by subscript "1" and using (G_0, B_m) as parameters brings (65) to the form

$$\alpha = G_0 + G_1(G_0, B_m, \beta) \quad (67)$$

Thus the shell's equation depends strongly on G_0 and only weakly, through the correction term G_1 , on the second parameter B_m . If G_1 is ignored - or better, replaced by $\langle G_1 \rangle$, its average over B_m - then all particles characterized by some value of G_0 will stay close to the surface

$$\alpha = G_0 + \langle G_1(G_0, \beta) \rangle \quad (68)$$

and thus their drift shell will still be approximately characterized by the single parameter G_0 . All the preceding also holds if G_0 is replaced by $L(G_0)$; in particular, McIlwain's definition of L [McIlwain, 1961] is equivalent to

$$L(I, B_m) = a g_1^0 / G_0(I, B_m) \quad (69)$$

In McIlwain's work the above function is approximated in two steps, beginning with the definition of auxiliary functions

$$\begin{aligned} Y &= \ln (L^3 B_m / g_1^0 - 1) \\ X &= \ln \xi \quad \text{where} \quad \xi = I^3 B_m / g_1^0 \end{aligned} \quad (70)$$

The relation between X and Y is then approximated by a polynomial

$$Y = \sum_{n=0}^N a_n X^n \quad (71)$$

In the original derivation [McIlwain, 1961] sixth-order polynomials were used and different expansions were chosen for each of 5 ranges of X . Later on [McIlwain, 1966] an improved approximation was derived, with N increased to 9 and the number of ranges to 6, and an inverse expansion, expressing X in terms of Y , was also provided. A much simpler and fairly accurate approximation has been proposed by Hilton [1971], who matched the analytical behavior of L at the limits $I = 0$ and $B_m \rightarrow \infty$ with that of the approximation. His result is

$$L^3 B_m / g_1^0 = 1 + a_1 \xi^{1/3} + a_2 \xi^{2/3} + a_3 \xi \quad (72)$$

where

$$a_1 = 3 (2)^{1/2} / \pi \approx 1.3507$$

$$a_2 = \left[2 + (3)^{1/2} \ln(2 + 3^{1/2}) \right]^{-3}$$

$$\approx 0.0475455$$

$$a_3 = 0.456376$$

The relative inaccuracy $\Delta L/L$ of this approximation is 10^{-4} or less.

The L parameter is widely used in labeling field lines, in the following manner. Given a point $P = (x, y, z)$ in the earth's field (which is assumed to be given by the expansion (11) of the scalar potential), the integral I is derived for particles mirroring at P , by numerical integration. Using one of the formulas described earlier, $L(I, B_m)$ is derived for these particles and its value is regarded as a labeling parameter for the field line passing P . Of course, what has been derived here is an approximation to $a g_1^0 / \alpha(x, y, z)$ where α is the perturbed-dipole generalization of α_0 in (42-a) (in principle $a g_1^0 / \alpha$ could be chosen as a new Euler potential α' , but the form of β' is then complicated). The derivation of this approximation is encumbered by the need for a numerical integration expressing I , although this integration has been simplified - as far as computer use is concerned - by G. Kluge (ESRO Internal Note 66, October 1970) who first transformed it by inversion with respect to the origin.

This correspondence between L and α works both ways and Stern [1968] has used approximations of α for perturbed dipole fields to obtain approximations to $L(I, B_m)$, avoiding the need for deriving I by line integration. This method is already implicit in the work of Pennington [1961, 1967] who derived the equations of drift shells in a perturbed dipole field by a straightforward perturbation method.

The L parameter can be generalized to any near-axisymmetric ^{poloidal} magnetic field : if (65) represents the family of drift shells in such a field, it is only necessary to derive the corresponding equation (66) for the underlying axisymmetric field and then choose G_0 or some function of G_0 as the L parameter.

Such an approach was applied by Stern [1968] to the 3-parameter model of the magnetosphere discussed in the preceding section. At moderate distances from the origin this field may be viewed as a perturbation of the axisymmetrical field given by

$$\begin{aligned} \underline{B} &= - \nabla \gamma \\ \gamma &= a \left[g_1^0(a/r) + \bar{g}_1^0(r/a)^2 \right] \cos \theta \end{aligned} \quad (73)$$

By a perturbation method Stern derived an approximate form for the equation corresponding to (66) in this field

$$\alpha = G'_0(I, B_m)$$

and denoted $L' = a g_1^0/G'$ as the generalized L -parameter. In principle the same generalization should be feasible for taking into account the axisymmetrical ring current of the earth, for treatment of particle motion in slightly asymmetrical mirror machines and for similar applications.

A number of quantities related to L are in general use. Among the best known of these is the invariant latitude Λ : given a point with some value of L , its "invariant latitude" satisfies

$$\cos^2 \Lambda = 1/L \quad (74)$$

The underlying idea is that if all points in the perturbed field were to be mapped to a dipole field in a way that L (or α) were conserved, then Λ would be the latitude at which the field line through the point met the earth's surface. A similar generalization for the longitude is provided by the magnetic longitude [McIlwain, 1966] .

(5) LOCAL REPRESENTATIONS

It is sometimes required to represent \underline{B} in the immediate neighborhood of some given point P without regard to the field's configuration in the rest of space. Two important cases in which this happens are in the guiding center motion of charged particles and in plasma dynamics near neutral points: it is useful in such cases to introduce so-called local representations of \underline{B} which are valid only in a restricted neighborhood of P .

The most common local representation - and the only one discussed here - is the Taylor expansion

$$\underline{B}(\underline{r}) = \underline{B}_0 + \underline{r} \cdot \nabla \underline{B}_0 + \frac{1}{2} \underline{r} \underline{r} : \nabla \nabla \underline{B}_0 + \dots \quad (75)$$

where all quantities with subscript zero are evaluated at the point P , regarded as the origin for \underline{r} . Such a representation is fully equivalent to (32) and is best handled by the methods developed in section (3-c), i.e. by expressing \underline{B} in the form

$$\underline{B} = \nabla \times \underline{r} \psi_1 + \nabla \times \nabla \times \underline{r} \psi_2 \quad (76)$$

and expanding ψ_1 and ψ_2 in polynomials in (x, y, z) . Let $\psi_i^{(j)}$ denote the sum of those terms in the expansion of ψ_i which form homogeneous polynomials of degree j in (x, y, z) . Then it is easily seen that

$$\begin{aligned} \underline{B}_0 &= \nabla \times \nabla \times \underline{r} \psi_2^{(1)} \\ \underline{r} \cdot \nabla \underline{B}_0 &= \nabla \times \nabla \times \underline{r} \psi_2^{(2)} + \nabla \times \underline{r} \psi_1^{(1)} \\ \frac{1}{2} \underline{r} \underline{r} : \nabla \nabla \underline{B}_0 &= \nabla \times \nabla \times \underline{r} \psi_3^{(3)} + \nabla \times \underline{r} \psi_1^{(2)} \end{aligned} \quad (77)$$

and so forth, with the degree of the terms on both sides increasing

by unity with each succeeding line. The coefficients on either side are not in general independent. On the left side the terms are related by virtue of the condition $\nabla \cdot \underline{B} = 0$; for instance, $\nabla \underline{B}_0$ is required to have zero trace, so that only 8 of its coefficients can be independently specified. On the right any combination of terms which depends on r alone does not contribute to the field and should be eliminated by the methods of section (3-c). The vector potential follows at once from (77); it should be noted that no simple relation exists between the expansion (75) of \underline{B} and a similar one for \underline{A} .

In principle [Morse and Feshbach, 1953] an alternative local expansion is possible in which \underline{r} in (76) is replaced by the constant unit vector $\hat{\underline{z}}$:

$$\underline{B} = \nabla \times \underline{\Phi}_1 \hat{\underline{z}} + \nabla \times \nabla \times \underline{\Phi}_2 \hat{\underline{z}} \quad (78)$$

This, however, introduces a considerable ambiguity. In (76) no component of \underline{B} can be both poloidal and toroidal, so that the assignment of such components to Ψ_1 and Ψ_2 is never arbitrary. The analogous statement does not hold for (78) and because of this, except perhaps for special cases, this expansion is not recommended.

Local Euler potentials are similarly expressed as polynomials of ascending orders

$$\begin{aligned} \alpha &= \alpha^{(1)} + \alpha^{(2)} + \dots \\ \beta &= \beta^{(1)} + \beta^{(2)} + \dots \end{aligned} \quad (79)$$

with

$$\begin{aligned} \underline{B}_0 &= \nabla \alpha^{(1)} \times \nabla \beta^{(1)} \\ \underline{r} \cdot \nabla \underline{B}_0 &= \nabla \alpha^{(1)} \times \nabla \beta^{(2)} + \nabla \alpha^{(2)} \times \nabla \beta^{(1)} \end{aligned} \quad (80)$$

and so forth. The calculation here is greatly simplified if the coordinate

axes are suitably chosen, with the z axis along \underline{B}_0 and

$$\alpha^{(1)} = x \qquad \beta^{(1)} = B_0 y$$

If then

$$\begin{aligned} \alpha^{(2)} &= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + (a'_{12} - 2b_{22})xy + a_{13}xz + a_{23}yz \\ \beta^{(2)} &= b_{11}x^2 + b_{22}y^2 + b_{33}z^2 + (b'_{12} - 2a_{11})xy + b_{13}xz + b_{23}yz \end{aligned} \quad (81)$$

it is found that the 4 coefficients a_{11} , a_{22} , b_{11} and b_{22} affect only orders higher than the first in (77), while the remaining 8 coefficients are uniquely related to the 8 independent terms which determine $\nabla \underline{B}_0$.

An interesting local representation, equivalent to (75) up to the order of $\nabla \underline{B}_0$, has been described by Siambis and Northrop [1966]. At a point P in the field let the unit vectors $(\hat{\underline{L}}, \hat{\underline{M}}, \hat{\underline{N}})$ form a right-handed orthogonal system with the origin at P , $\hat{\underline{L}}$ parallel to \underline{B} and $\hat{\underline{M}}$ pointing towards the center of curvature of the field line through P . In this system the components of $\nabla \underline{B}_0$ can be expressed in terms of curvature and shear coefficients (loc. cit., appendix) while the coefficients themselves can be expressed in terms of the spatial variation of the basic unit vectors following a shift in P . Some applications of such a system of local vectors exist in plasma theory, e.g. in the work of Wilson [1970].

The Taylor expansion (75) forms a focal point of the theory of neutral points [Dungey, 1953, 1963]. A first-order neutral point exists at P of \underline{B}_0 vanishes but $\nabla \underline{B}_0$ does not, so that to lowest order, in the vicinity of P

$$\underline{B} = \underline{r} \cdot \nabla \underline{B}_0 \quad (82)$$

Even though \underline{B} vanishes at P there may exist singular field lines which pass through P and at sufficiently small distances from P such field lines can then be approximated by straight lines. In the region of interest the magnetic field \underline{B} depends linearly on (x, y, z) as measured from P and therefore, if $\underline{\xi}$ is the radial distance measured from P along one of the singular lines, its value on that line should satisfy (to lowest order)

$$\underline{B} = \lambda \underline{\xi} \quad (83)$$

where λ is some constant. By (82)

$$\lambda \underline{\xi} = \underline{\xi} \cdot \nabla \underline{B}_0 \quad (84)$$

and therefore λ must be a real eigenvalue of $\nabla \underline{B}_0$. There may exist either 1 or 3 such eigenvalues; accordingly, there will exist 1 or 3 such singular field lines and P will be classified as an O-type or X-type neutral point, so named because field line configurations near P resemble either the letter O or the letter X (Figure 2).

A great amount of theoretical work and of interest has focused on X-type neutral points, since they play a central role in processes which change the topology of magnetic field lines; for a review the reader is referred to the work of Vasyliunas [1975]. Surprisingly, only limited experimental work exists on the subject [e.g. Baum et al., 1973], nor has there been much interest in magnetic field configurations in space with O-type neutral points.

AFTERWORD

This is a draft of a review article and additional relevant material may be added to the final version. The author welcomes any suggestions for such additions as well as comments of a general nature.

Two additions which will be incorporated concern the description of two extraterrestrial magnetic fields by harmonic expansions similar to eq. (11). Altschuler and Newkirk [1969] applied such an expansion to the magnetic field of the sun and their work was extended by Schatten [1971 ; Schatten and Howell, 1971] . In addition, Acuna and Ness [1975] extracted a model of the external field of Jupiter from the observations of Pioneer 11, which passed close enough to the planet to allow the larger non-dipole harmonic terms to be estimated.

Another such addition concerns the work of Voigt [1972] , who developed a general class of current-free magnetospheric models. In Voigt's approach the bounding surface (i.e. magnetopause) is deduced from observations and serves as a boundary condition to which an expansion of γ is then fitted.

CAPTIONS TO FIGURES

Figure 1 - Lines of constant electrical potential $\phi(\alpha, \beta)$ (solid) and of constant conjugate potential $\psi(\alpha, \beta)$ (dashed) for a dipole magnetic field in the equatorial plane, using a simple analytical model of the earth's electric field.

Figures 2 - The behavior of magnetic field lines near an X-type (2-a) and an O-type (2-b) neutral point. In Figure (2-b) the pattern below the neutral point mirrors the one above it and the main axes need not be orthogonal.

REFERENCES

- Acuna, M.H. and N.F. Ness, The complex magnetic field of Jupiter, submitted to Science, 1975; GSFC X-690-75-42, February 1975.
- Akasofu, S.-I. and S. Chapman, The ring current, geomagnetic disturbance, and the Van Allen Belt, J. Geophys. Res. 66, 1321-1350, 1971; correction J. Geophys. Res. 72, 445-446, 1967.
- Altschuler, M.D. and G. Newkirk, Jr., Magnetic fields and the structure of the solar corona, Solar Physics 9, 131-149, 1969.
- Babcock, H.W., The topology of the Sun's magnetic field and the 22-year cycle, Ap. J. 133, 572-587, 1961.
- Backus, G.E., Application of a non-linear boundary value problem for Laplace's equation to gravity and geomagnetic intensity surveys, Quart. J. Mech. Appl. Math. 21, 195-221, 1968.
- Backus, G.E., Non-uniqueness of the external geomagnetic field determined by surface intensity measurements, J. Geophys. Res. 75, 6339-6341, 1970.
- Backus, G.E., Determination of the external geomagnetic field from intensity measurements, Geophys. Res. Lett., 1, 21, 1974.
- Barish, F.D., R. A. Smith and D. P. Stern, Models of the Jovian magnetosphere (abstract), EOS, 55, 694, 1974.
- Bartels, J., The eccentric dipole approximating the Earth's magnetic field, Terrestrial Magnetism and Atmospheric Electricity 41, 225-250, 1936.
- Baum, P.J., A. Bratenahl and R.S. White, X-ray and electron spectra from the double inverse pinch device, Phys. Fluids 16, 226-230, 1973.
- Bonnevier, B., R. Boström and G. Rostocker, A three dimensional model current system for polar magnetic substorms, J. Geophys. Res. 75, 107-122, 1970.

- Burch, J.L. and R. W. Janetzke, Coordinate transformations for studies of interactions between interplanetary and geomagnetic fields, GSFC X-621-74-179, April 1974.
- Cain, J.C., S.J. Hendricks, R.A. Langel and V.W. Hudson, A proposed model for the international geomagnetic reference field - 1965, J. Geomagn. Geoelect. 19, 335-355, 1967.
- Cain, J.C., Geomagnetic models from satellite surveys, Rev. Geophys. and Space Phys. 9, 259-273, 1971.
- Chapman, S., Geomagnetic nomenclature, J. Geophys. Res. 68, 1174, 1963.
- Chapman, S. and V.C.A. Ferraro, A new theory of magnetic storms, Terrest. Magnetism and Atmospheric Elec. 36, 77-97 and 171-186, 1931.
- Chapman, S. and J. Bartels, Geomagnetism, Oxford Univ. Press, 1940.
- Chen, A.J. and D.P. Stern, Adiabatic hamiltonian of charged particle motion in a dipole field, J. Geophys. Res. 80, 690-693, 1975.
- Crooker, N.U. and G.L. Siscoe, Model geomagnetic disturbance from asymmetric ring current particles, J. Geophys. Res. 79, 589-603, 1974.
- Dungey, J.W., Conditions for the occurrence of electrical discharges in astrophysical systems, Phil. Mag., Series 7, 44, 725-738, 1953.
- Dungey, J.W., Cosmic Electrodynamics, Cambridge Univ. Press, 1958.
- Dungey, J.W., The structure of the exosphere, or, adventures in velocity space, in Geophysics: The Earth's Environment, (Proceedings of the 1962 Les Houches Summer School), Gordon and Breach 1963, p. 502-552.

- Elsasser, W.M., Induction effects in terrestrial magnetism, Part I:
theory, Phys. Rev. 69, 106-116, 1946.
- Elsasser, W.M., Induction effects in terrestrial magnetism, Part II:
the secular variation, Phys. Rev. 70, 202-212, 1947.
- Elsasser, W.M., Induction effects in terrestrial magnetism, Part III:
electric modes, Phys. Rev. 72, 821-833, 1947.
- Elsasser, W.M., Hydrodynamic dynamo theory, Rev. Mod. Phys. 28, 135-163,
1956.
- Fairfield, D.H. and G.D. Mead, Magnetospheric mapping with a quantitative
geomagnetic field model, J. Geophys. Res. 80, 535-542, 1975.
- Forbes, T.G. and T.W. Speiser, Mathematical models of the open
magnetosphere: application to dayside auroras, J. Geophys. Res.
78, 7542-7551, 1971.
- Gardner, C., Adiabatic invariants of classical periodic systems, Phys.
Rev. 115, 791-794, 1959.
- Hansen, W.W., A new type of expansion in radiation problems, Phys. Rev.
47, 139-143, 1935.
- Hassitt, A., Coordinates in the earth's magnetic field, Publication
UCSD-SP-67-4 of the University of California at San Diego,
June 1967.
- Heppner, J.P. The world magnetic survey, Space Science Reviews 2, 315-
354, 1963.
- Hilton, H.H., L parameter, a new approximation, J. Geophys. Res. 76,
6952-6954, 1971.

- Hilton, H.H. and M. Schulz, Geomagnetic potential in offset dipole coordinates, J. Geophys. Res. 78, 2324-2330, 1973.
- Hones, E.W. Jr., Motions of charged particles trapped in the earth's magnetosphere, J. Geophys. Res. 68, 1209-1219, 1963.
- Hurwitz, L. and D.G. Knapp, Inherent vector discrepancies in geomagnetic main field models based on Scalar F, J. Geophys. Res. 79, 3009-3013, 1974.
- IAGA Commission 2, Working Group 4 International Geomagnetic Reference Field 1965.0, J. Geophys. Res. 74, 4407-4408, 1969.
- Kaula, W.M., An Introduction to Planetary Physics, 490 pp., John Wiley and Sons, 1968.
- Kendall, P.C., S. Chapman, S.-I. Akasofu and P.N. Swartztrauber, The computation of the magnetic field of any axisymmetric current distribution - with magnetospheric applications, Geophys. J. Roy. Astr. Soc. 11, 349-364, 1966.
- Kono, M., Uniqueness of the spherical harmonic analysis of the geomagnetic field based on inclination and declination data
- Kosik, J.C., Mouvement des particules chargees dans un modele analytique de la magnetosphere de Mead, Annales de Geophysique 27, 11-26, 1971.
- Kosik, J.C., L'environnement en particules energetiques aux altitudes synchrones au cours des periodes geomagnetiques calmes, Annales de Geophysique 27, 175-180, 1971.

- James, R.W., Transformation of spherical harmonics under change of reference frame, Geophys. J. of Roy. Astr. Soc. 17, 305-316, 1969.
- Leighton, R.B., A magneto-kinematic model of the solar cycle, Ap. J. 156, 1-26, 1969.
- Longmire, C.L., Elementary Plasma Physics, 296 pp., Interscience, New York, 1963.
- McDonald, K.L. and R.H. Gunst, An analysis of the earth's magnetic field from 1835 to 1965, ESSA technical report IER 46 - IES 1, July 1967.
- McIlwain, C.E., Coordinates for mapping the distribution of magnetically trapped particles, J. Geophys. Res. 66, 3681-3691, 1961.
- McIlwain, C.E., Magnetic coordinates, Radiation Trapped in the Earth's Magnetic Field (B. McCormac Ed.), D. Reidel, 1966; also Space Sci. Rev. 5, 585-598, 1966.
- Mead, G.D., Deformation of the geomagnetic field by the solar wind, J. Geophys. Res. 69, 1181-1195, 1964.
- Mead, G.D. and D.H. Fairfield, A quantitative magnetospheric model derived from spacecraft magnetometer data, J. Geophys. Res. 80, 523-534, 1975.
- Midgley, J.E., Perturbation of the geomagnetic field, a spherical harmonic expansion, J. Geophys. Res. 69, 1197-1200, 1964.
- Mie, Gustav, Beiträge zur optik trüber medien, speziell kolloidaler metallösungen, Ann. Phys. (series 4) 25, 377-445, 1908.
- Morse, P.M. and H. Feshbach, Methods of Theoretical Physics, McGraw Hill, 1953.

- Northrop, T.G., The Adiabatic Motion of Charged Particles, 109 pp,
John Wiley and Sons (Interscience) 1963.
- Northrop, T.G. and E. Teller, Stability of the adiabatic motion of charged
particles in the earth's field, Phys. Rev. 117, 210-225, 1960.
- Olson, W.P., The shape of the tilted magnetopause , J. Geophys. Res.
74, 5642-5651, 1969.
- Olson, W.P., A model of the distributed magnetospheric currents,
J. Geophys. Res. 79, 3731-3738, 1974.
- Olson, W.P. and K.A. Pfizter, A quantitative model of the magnetospheric
magnetic field, J. Geophys. Res. 79, 3739-3748, 1974.
- Parker, E.N., Hydrodynamic dynamo models, Ap. J. 122, 293-314, 1955.
- Parker, E.N., Geomagnetic fluctuations and the form of the outer zone
of the Van Allen radiation belt, J. Geophys. Res. 76, 3117-3130,
1960.
- Pennington, R.H., Equation of a charged particle shell in a perturbed
dipole field, J. Geophys. Res. 66, 709-712, 1961.
- Pennington, R.H., Derivation of geomagnetic shell equations, Air Force
Weapons Lab. Rept. AFWL-TR-66-152, 1967.
- Russbridge, M.G., Non-adiabatic charged particle motion near a magnetic
field zero line, Plasma Physics 13, 977-987, 1971.
- Russell, C.T., Geophysical coordinate transformations, Cosmic Electrodynamics
2, 184-196, 1971.
- Schatten, K.H., Current sheet magnetic model for the solar corona, Cosmic
Electrodynamics 2, 232-245, 1971.
- Schatten, K.H. and D.R. Howell, Corona: a program for computing magnetic
field lines in the quiet solar corona, GSFC X-692-71-336, August
1971.

Schield, M.A., Pressure balance between solar wind and magnetosphere,
J. Geophys. Res. 74, 1275-1286, 1969 (appendix).

Senior, T.B.A., A note on Hansen's vector wave functions, Canad.
J. of Phys. 38, 1702-1705, 1960.

Siambis, J.G. and T.G. Northrop, Magnetic field geometry and the adiabatic
invariants of particle motion, Phys. Fluids 9, 2001-2005, 1966.

Slater, J.C., Quantum Theory of Atomic Structure, vol. 2, appendix
23, McGraw Hill, New York 1960.

Smith, E.J., L. Davis, Jr., D.E. Jones, P.J. Coleman, Jr., D.S. Colburn,
P. Dyal, C.P. Sonett and A.M.A. Frandsen, The planetary magnetic
field and magnetosphere of Jupiter: Pioneer 10, J. Geophys. Res.
79, 3501-3513, 1974.

Stacey, F.D., Physics of the Earth, John Wiley and Sons, 1969.

Stern, D.P., The vector potential and motion of charged particles in
axisymmetric magnetic fields, J. Geophys. Res. 69, 2715-2719,
1964.

Stern, D.P. Geomagnetic Euler potentials, J. Geophys. Res. 72, 3995-
4005, 1967.

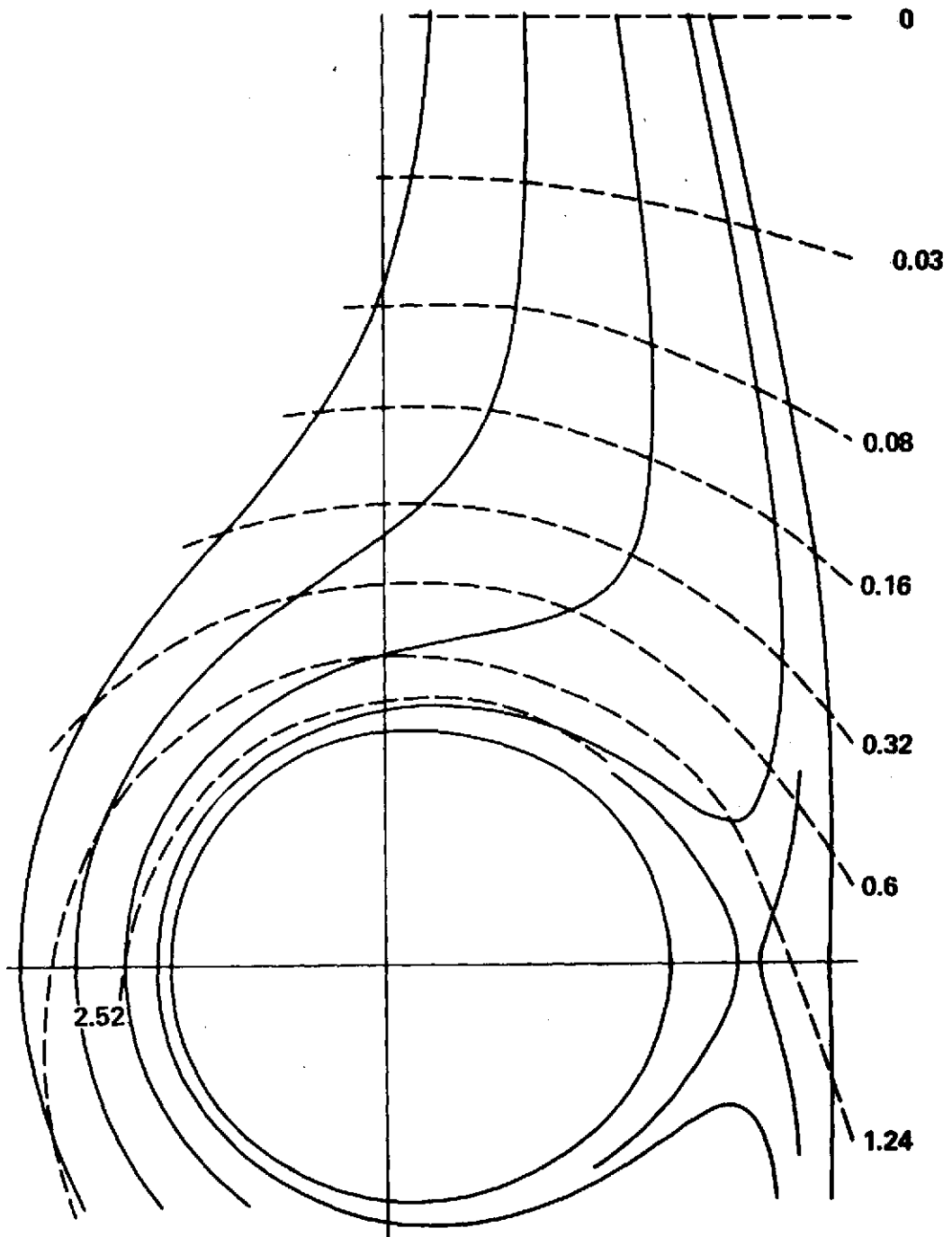
Stern, D.P., Euler potentials and geomagnetic drift shells, J. Geophys.
Res. 73, 4373-4378, 1968.

Stern, D.P., Euler potentials, Amer. J. Phys. 38, 494-501, 1970.

Stern, D.P., Geomagnetic potential in dipole coordinates, J. Geophys.
Res. 76, 257-258, 1971.

- Stern, D.P., Models of the earth's electric field, GSFC X-602-74-159, May 1974.
- Stern, D.P., Charged particle motions in a magnetic field which reduce to motions in a potential, Amer. J. Phys. 43, 1975.
- Stern, D.P. and J.H. Bredekamp, Error enhancement in geomagnetic models derived from scalar data, J. Geophys. Res. 80, 1975.
- Sweet, P.A., The effect of turbulence on a magnetic field, Monthly Notices Roy. Astr. Soc. 110, 69-83, 1950.
- Taylor, H.E. and E.W. Hones, Jr., Adiabatic motion of auroral particles in a model of the electric and magnetic fields surrounding the earth, J. Geophys. Res. 70, 3605-3628, 1965.
- Vasyliunas, V.M., Theoretical models of magnetic field line merging, Rev. Geophys. and Space Phys. (to be published, 1975).
- Vestine, E.H., The survey of the geomagnetic field in space, Transact. A.G.U. 41, 4-21, 1960.
- Voigt, G.-H., A 3-dimensional, analytical magnetospheric model with defined magnetopause, Z. f. Geophysik 38, 319-346, 1972.
- Williams, D.J. and G.D. Mead, Nightside magnetosphere configuration obtained from trapped electrons at 1100 kilometers, J. Geophys. Res. 70, 3017-3029, 1965.
- Wilson, G.E., Comparison of three sets of higher order adiabatic plasma equations, Phys. Fluids 13, 1372-1379, 1970.
- Zmuda, A.J., The geomagnetic field and its harmonic description, Geomagnetism and Aeronomy 13, 1098-1109, 1973 (in Russian).

Figure 1



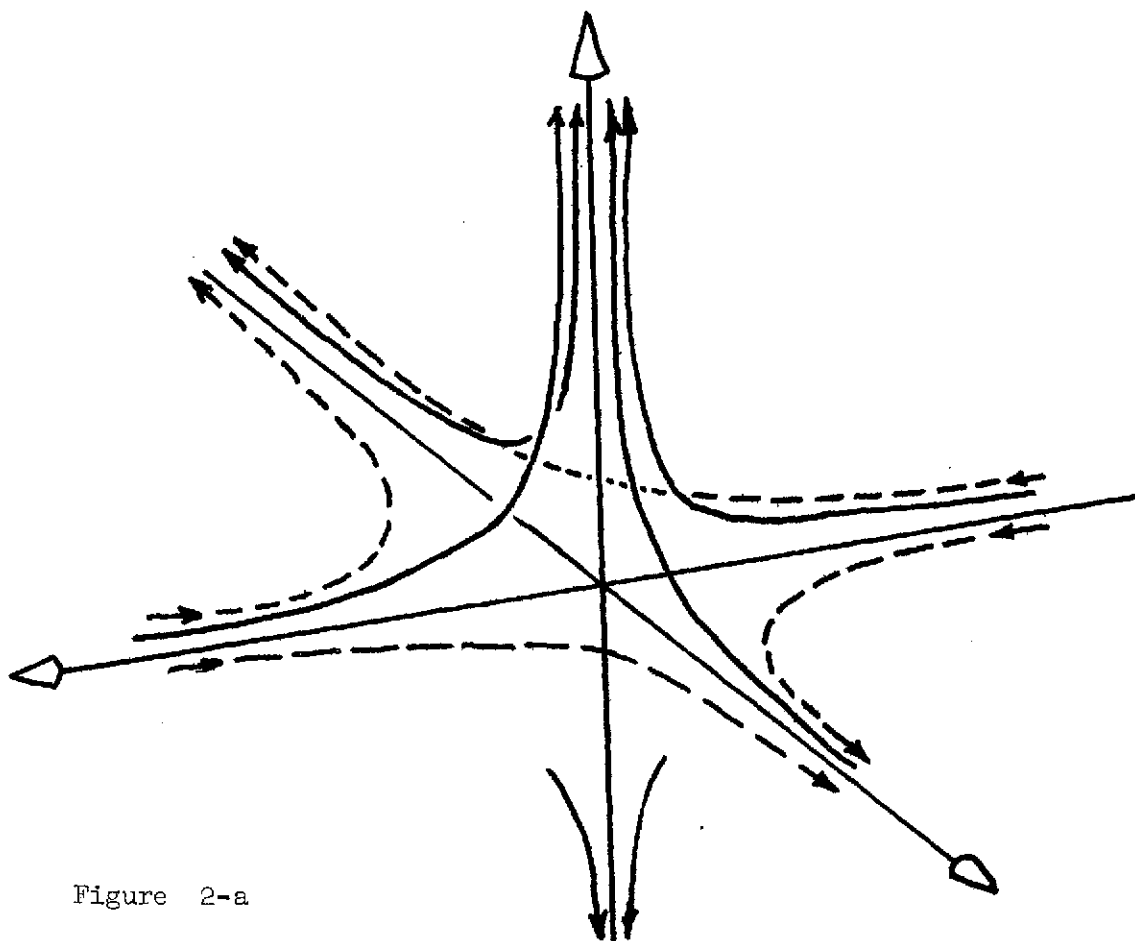


Figure 2-a

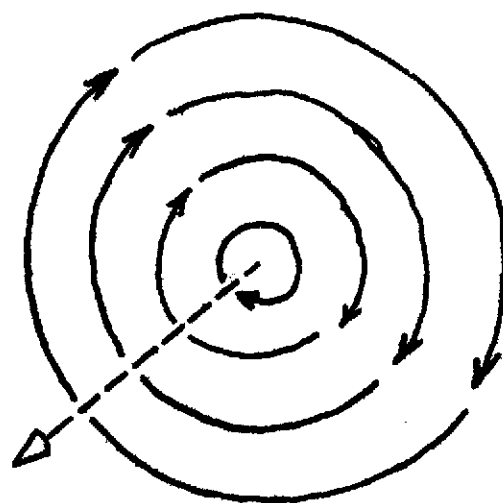


Figure 2-b